

Proof of László Fejes Tóth's zone conjecture

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Abstract

A zone of width ω on the unit sphere is the set of points within spherical distance $\omega/2$ of a given great circle. We show that the total width of any collection of zones covering the unit sphere is at least π , answering a question of Fejes Tóth from 1973.

1 Introduction

A *plank* (or *slab*, or *strip*) of width w is a part of the d -dimensional Euclidean space \mathbb{R}^d that lies between two parallel hyperplanes at distance w . Given a convex body C , its *width* is the smallest w such that a plank of width w covers C . In 1932, in the context of “degree of equivalence of polygons”, Tarski wrote in [Tar32]¹, and we quote from its English translation [MMS14, Chapter 7.4],

The width of the narrowest strip covering a plane figure F we shall call the *width of figure* F , and we shall denote it by the symbol $\omega(F)$

A. *If a figure F contains in itself, as a part, a disk with diameter equal to the width of the figure (for example, if figure F is a disk or a parallelogram) and if, moreover, we subdivide this figure into any n parts C_1, C_2, \dots, C_n , then*

$$\omega(F) \leq \omega(C_1) + \omega(C_2) + \dots + \omega(C_n).$$

Proof. For the case in which figure F is a disk, the proof is an almost word-for-word repetition of the proof of lemma I in the cited article by Moese. For the general case, ...

Here “the cited article by Moese” refers to [Moe32]¹. For a long period of time, the authors of the current paper mistakenly overlooked the contribution of Moese.

The following conjecture, which seems to first appear in [Ban50], was attributed to Tarski.

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¹See [MMS14, Chapter 7.3, 7.4] for English translations of [Moe32] and [Tar32].

Tarski's plank problem. If a convex body of width w is covered by a collection of planks in \mathbb{R}^d , then the total width of the planks is at least w .

It took almost twenty years before Bang proved Tarski's conjecture in his memorable papers [Ban50, Ban51]. Generalizations and variants of Tarski's plank problem were considered in various directions. See [Bez13] for a recent survey on this topic.

A *zone* of width ω on the 2-dimensional unit sphere is defined as the set of points within spherical distance $\omega/2$ of a given great circle. Zones can be thought as the spherical analogue of planks. In 1973, Fejes Tóth [Tót73] conjectured that if n equal zones cover the sphere then their width is at least $\omega_n = \pi/n$. Rosta [Ros72] and Linhart [Lin74] proved the special case of 3 and 4 zones respectively, and they both showed that the unique optimal configuration consists of zones whose central great circles pass through two antipodal points. Lower bounds for ω_n were established by Fodor, Víg and Zarnócz [FVZ16]. Fejes Tóth also formulated the generalized conjecture for any set of zones covering the unit sphere.

Fejes Tóth's zone conjecture. The total width of any set of zones covering the sphere is at least π .

In this paper, we completely resolve this conjecture and generalize it for the d -dimensional unit sphere S^d . Hereafter, all d -spheres are embedded in \mathbb{R}^{d+1} and centered at the origin. For us, a *great sphere* is the intersection of a d -sphere and a hyperplane passing through the origin. This extends the notion of a great circle on a 2-dimensional sphere. Given a great sphere C , a zone P on S^d can be defined similarly as on S^2 , and the *central hyperplane* of P is the hyperplane through C .

Theorem 1. *The total width of any collection of zones P_1, \dots, P_n covering the unit d -sphere is at least π . For all $i \in [n]$, let $2\alpha_i$ be the width of P_i , and let l_i be the line through the origin perpendicular to the central hyperplane of P_i . Equality holds if and only if after reordering the zones, l_1, \dots, l_n are coplanar lines in clockwise order such that the angle l_i and l_{i+1} equals to $\alpha_i + \alpha_{i+1}$ for all $i \in [n]$ under the convention that $l_{n+1} = l_1$ and $\alpha_{n+1} = \alpha_1$.*

Our result is also connected to covering of a non-separable family of homothetic copies of a convex body. The first result in this direction was due to Goodman and Goodman [GG45]. They proved the following theorem conjectured by Erdős.

Theorem 2. *Let the circles with radii r_1, \dots, r_n lie in a plane. If no line divides the circles into two non-empty sets without intersecting at least one circle, then the circles can be covered by a circle of radius $r = \sum_{i=1}^n r_i$.*

Its spherical relative follows immediately from Theorem 1 and projective duality. A *cap* of spherical radius r on the unit d -sphere is defined as the set of points within spherical distance r of a given point on the sphere.

Corollary 3. *Let the caps with spherical radii r_1, \dots, r_n lie on the unit d -sphere. If every great sphere intersects at least one cap, then the total radius $\sum_{i=1}^n r_i$ is at least $\pi/2$.*

The proof of Theorem 1 will be given in the next section. In Section 3, we state a few more corollaries, one of which partially confirms the following question in [Tót73]:

Fejes Tóth's zone problem. A convex spherical domain is a domain any two points of which can be joined by an arc of a great circle lying in the domain. Is it true that if a convex spherical domain is covered with a set of zones of total width w , then it can be covered with one zone of width w ?

For a list of related open problems, we refer the interested readers to [BMP05, Chapter 3.4].

2 Proof of Theorem 1

Our proof is inspired by Bang [Ban51], Goodman and Goodman [GG45]. Roughly speaking, the idea of the proof is to show that

1. either a particular discrete subset not fully covered by the convex hulls (taken in \mathbb{R}^{d+1}) of the zones is contained in the unit ball;
2. or several zones can be covered and replaced by one zone without increasing the total width.

In this section, we denote points also by vectors leading to them from the origin. We need the following technical lemmas, the first of which is in the spirit of Theorem 2.

Lemma 4. *Suppose the total spherical radius of a collection of caps D_1, \dots, D_n on S^d is at most $\pi/2$. For all $i \in [n]$, let α_i be the spherical radius of D_i , let \mathbf{u}_i be the center of D_i , and set $\mathbf{w}_i := \sin \alpha_i \mathbf{u}_i$. Denote $\mathbf{w} := \sum_{i=1}^n \mathbf{w}_i$ and $\alpha := \sum_{i=1}^n \alpha_i$. If*

$$|\mathbf{w}| \geq \sin \alpha \text{ and } |\mathbf{w} - \mathbf{w}_i| \leq \sin(\alpha - \alpha_i) \text{ for all } i \in [n], \quad (1)$$

then a cap with spherical radius $\leq \alpha$ can cover $\cup_{i=1}^n D_i$. Moreover, a cap with spherical radius $< \alpha$ can cover $\cup_{i=1}^n D_i$ unless

$$|\mathbf{w}| = \sin \alpha \text{ and } |\mathbf{w} - \mathbf{w}_i| = \sin(\alpha - \alpha_i) \text{ for some } i \in [n]. \quad (2)$$

Remark 1. *In case that (2) holds, by the law of sines, $\angle(\mathbf{w}, \mathbf{w}_i) = \alpha - \alpha_i$ (see Figure 1).*

Proof. We claim that the cap D centered at $\mathbf{u} := \mathbf{w}/|\mathbf{w}|$ with spherical radius α covers $\cup_{i=1}^n D_i$. Suppose on the contrary that say D_i is not covered by D . Clearly, the angle between \mathbf{u} and \mathbf{u}_i is strictly greater than $\alpha' := \alpha - \alpha_i$. By the law of cosines and the monotonicity of cosine on $(0, \pi)$, we get

$$\begin{aligned} |\mathbf{w} - \mathbf{w}_i|^2 &= |\mathbf{w}|^2 + |\mathbf{w}_i|^2 - 2|\mathbf{w}||\mathbf{w}_i|\cos\angle(\mathbf{u}, \mathbf{u}_i) > |\mathbf{w}|^2 + \sin^2 \alpha_i - 2|\mathbf{w}|\sin \alpha_i \cos \alpha' \\ &= (|\mathbf{w}| - \sin \alpha_i \cos \alpha')^2 + \sin^2 \alpha_i \sin^2 \alpha'. \end{aligned} \quad (3)$$

Notice that $|\mathbf{w}| \geq \sin \alpha = \sin \alpha_i \cos \alpha' + \cos \alpha_i \sin \alpha' > \sin \alpha_i \cos \alpha'$. The right hand side of (3) is at least

$$(\sin \alpha - \sin \alpha_i \cos \alpha')^2 + \sin^2 \alpha_i \sin^2 \alpha' = \cos^2 \alpha_i \sin^2 \alpha' + \sin^2 \alpha_i \sin^2 \alpha' = \sin^2 \alpha'.$$

Therefore $|\mathbf{w} - \mathbf{w}_i| > \sin \alpha' = \sin(\alpha - \alpha_i)$ contradicts (1).

Clearly if $|\mathbf{w}| > \sin \alpha$ or $|\mathbf{w} - \mathbf{w}_i| < \sin(\alpha - \alpha_i)$, then the cap centered at \mathbf{u} with radius $\alpha - \varepsilon_i$ covers D_i for some $\varepsilon_i > 0$. Hence if $|\mathbf{w}| > \sin \alpha$ or $|\mathbf{w} - \mathbf{w}_i| < \sin(\alpha - \alpha_i)$ for all $i \in [n]$, then the cap centered at \mathbf{u} with radius $\alpha - \min \{\varepsilon_i : i \in [n]\}$ covers $\cup_{i=1}^n D_i$. \square

Lemma 5. *Suppose the total spherical radius of 3 caps D_1, D_2, D_3 is $\alpha \leq \pi/2$. For all $i \in [3]$, let α_i be the spherical radius of D_i , let \mathbf{u}_i be the center of D_i , and set $\mathbf{w}_i := \sin \alpha_i \mathbf{u}_i$. Suppose that*

$$|\mathbf{w}_1 + \mathbf{w}_2| = \sin(\alpha_1 + \alpha_2) \text{ and } |\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3| = \sin \alpha.$$

If $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are not coplanar, then a cap with spherical radius $< \alpha$ can cover $D_1 \cup D_2 \cup D_3$.

Proof. Denote $\mathbf{w}_{12} := \mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{w}_{123} := \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$. By the law of cosines and trigonometric identities, we obtain that

$$\cos \angle(\mathbf{w}_{12}, \mathbf{w}_1) = \frac{|\mathbf{w}_{12}|^2 + |\mathbf{w}_1|^2 - |\mathbf{w}_2|^2}{2|\mathbf{w}_{12}||\mathbf{w}_1|} = \frac{\sin^2(\alpha_1 + \alpha_2) + \sin^2 \alpha_1 - \sin^2 \alpha_2}{2 \sin(\alpha_1 + \alpha_2) \sin \alpha_1} = \cos \alpha_2.$$

Therefore $\angle(\mathbf{w}_{12}, \mathbf{w}_1) = \alpha_2$. Similarly, one can compute

$$\angle(\mathbf{w}_{12}, \mathbf{w}_2) = \alpha_1, \angle(\mathbf{w}_{123}, \mathbf{w}_{12}) = \alpha_3, \angle(\mathbf{w}_{123}, \mathbf{w}_3) = \alpha_1 + \alpha_2.$$

Suppose that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are not coplanar. Set $\mathbf{w} = \mathbf{w}(\varepsilon) := \mathbf{w}_{123} + \varepsilon \mathbf{w}_3$. We claim that cap $D = D(\varepsilon)$ centered at $\mathbf{w}/|\mathbf{w}|$ with spherical radius $< \alpha$ can cover $D_1 \cup D_2 \cup D_3$ for some $\varepsilon > 0$. Because $\mathbf{w}_1, \mathbf{w}_{12}, \mathbf{w}_{123}$ are not coplanar, by the spherical triangle inequality, we obtain

$$\delta_1 := \angle(\mathbf{w}_{123}, \mathbf{w}_{12}) + \angle(\mathbf{w}_{12}, \mathbf{w}_1) - \angle(\mathbf{w}_{123}, \mathbf{w}_1) = \alpha_2 + \alpha_3 - \angle(\mathbf{w}_{123}, \mathbf{w}_1) > 0.$$

Therefore there exists $c_1 \in \mathbb{R}$ such that

$$\angle(\mathbf{w}, \mathbf{w}_1) = \angle(\mathbf{w}_{123} + \varepsilon \mathbf{w}_3, \mathbf{w}_1) = \angle(\mathbf{w}_{123}, \mathbf{w}_1) + c_1 \varepsilon + O(\varepsilon^2) = \alpha_2 + \alpha_3 - \delta_1 + c_1 \varepsilon + O(\varepsilon^2). \quad (4)$$

Similarly, $\delta_2 := \alpha_1 + \alpha_3 - \angle(\mathbf{w}_{123}, \mathbf{w}_2) > 0$ and there exists $c_2 \in \mathbb{R}$ such that

$$\angle(\mathbf{w}, \mathbf{w}_2) = \alpha_1 + \alpha_3 - \delta_2 + c_2 \varepsilon + O(\varepsilon^2). \quad (5)$$

Since $\mathbf{w}_{123}, \mathbf{w}_{12}, \mathbf{w}_3$ are coplanar, but not collinear, there exists $c_3 > 0$ such that

$$\angle(\mathbf{w}, \mathbf{w}_3) = \angle(\mathbf{w}_{123} + \varepsilon \mathbf{w}_3, \mathbf{w}_3) = \angle(\mathbf{w}_{123}, \mathbf{w}_3) - c_3 \varepsilon + O(\varepsilon^2) = \alpha_1 + \alpha_2 - c_3 \varepsilon + O(\varepsilon^2), \quad (6)$$

In (4, 5, 6), one can take sufficiently small $\varepsilon > 0$ so that $\angle(\mathbf{w}, \mathbf{w}_i) < \alpha - \alpha_i$ for all $i \in [3]$. This proves the claim. \square

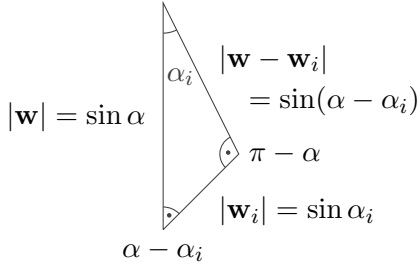


Figure 1

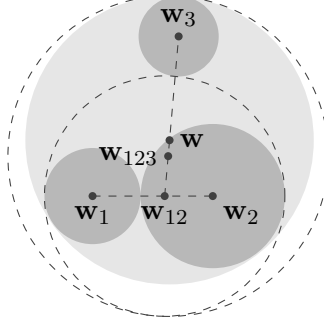


Figure 2

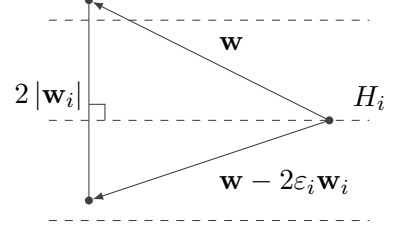


Figure 3

Remark 2. Because $\angle(\mathbf{w}_{123}, \mathbf{w}_1) < \alpha_2 + \alpha_3$, $\angle(\mathbf{w}_{123}, \mathbf{w}_2) < \alpha_1 + \alpha_3$ and $\angle(\mathbf{w}_{123}, \mathbf{w}_3) = \alpha_1 + \alpha_2$, the cap, say D_{123} , centered at $\mathbf{w}_{123}/|\mathbf{w}_{123}|$ with spherical radius $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ covers $\cup_{i=1}^3 D_i$. The idea of the proof of Lemma 5 is to move the center of D_{123} towards $\mathbf{w}_3/|\mathbf{w}_3|$ slightly and shrink its spherical radius properly so that the cap would still cover $\cup_{i=1}^3 D_i$ (see Figure 2).

Proof of Theorem 1. Suppose that the total width of a collection of zones P_1, \dots, P_n on S^d is less than π . Let $2\alpha_i$ be the width of P_i , let H_i be the central hyperplane of P_i , and let \mathbf{u}_i be a unit normal vector of H_i . Set $\mathbf{w}_i := \sin \alpha_i \mathbf{u}_i$ for all $i \in [n]$. Consider the set

$$L := \left\{ \sum_{i=1}^n \varepsilon_i \mathbf{w}_i : (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n \right\}.$$

Let $\mathbf{w} := \sum_{i=1}^n \varepsilon_i \mathbf{w}_i$ achieve the maximal norm among points of L . For every $i \in [n]$, using the fact that $|\mathbf{w}| \geq |\mathbf{w} - 2\varepsilon_i \mathbf{w}_i|$, one can easily see that the distance from \mathbf{w} to H_i is at least $|\mathbf{w}_i| = \sin \alpha_i$ (see Figure 3).

If $|\mathbf{w}| < 1$, then $\mathbf{w}/|\mathbf{w}| \notin \cup_{i=1}^n P_i$. Otherwise, $|\mathbf{w}| \geq 1$. Without loss of generality, we may assume that $\varepsilon_1 = \dots = \varepsilon_n = 1$. In particular, $|\sum_{i=1}^n \mathbf{w}_i| > \sin(\sum_{i=1}^n \alpha_i)$ because the total width $2\alpha_1 + \dots + 2\alpha_n < \pi$. Find the minimal set $I \subset [n]$ such that $|\sum_{i \in I} \mathbf{w}_i| > \sin(\sum_{i \in I} \alpha_i)$. For each $i \in I$, denote by D_i the cap centered at \mathbf{u}_i with spherical radius α_i . By the minimality of I , we can apply the first part of Lemma 4 to $\{D_i : i \in I\}$ to find a cap, say D , covering $\cup_{i \in I} D_i$ with radius $\leq \sum_{i \in I} \alpha_i$. Using the projective duality, one can check that the dual of D_i is P_i and the dual of D is a zone, say P , of width $\leq \sum_{i \in I} \alpha_i$. Moreover, because $\cup_{i \in I} D_i$ is covered by D and the projective duality preserves containment, $\cup_{i \in I} P_i$ is covered by P . Clearly, $|I| > 1$. Now we can reduce the number of zones by replacing $\{P_i : i \in I\}$ by P without increasing the total width, and we repeat the above argument for zones $\{P\} \cup \{P_i : i \in [n] \setminus I\}$.

Finally, we investigate the case when equality holds. Suppose that zones P_1, \dots, P_n of total width $2\alpha = \pi$ cover S^d . For all $i \in [n]$, let D_i be the dual cap of P_i . From the previous argument, we must have $|\mathbf{w}| = 1 = \sin \alpha$. Moreover, for any $I \subset [n]$, caps $\{D_i : i \in I\}$ cannot be covered by a cap of spherical radius $< \sum_{i \in I} \alpha_i$ since otherwise we can cover the sphere with planks of total width $< \pi$.

The second part of Lemma 4 shows that $|\mathbf{w} - \mathbf{w}_i| = \sin(\alpha - \alpha_i)$ for some $i \in [n]$. Without loss of generality, assume that $|\mathbf{w} - \mathbf{w}_n| = \sin(\alpha - \alpha_n)$. Again, apply the second part of Lemma 4 to D_1, \dots, D_{n-1} , it must be the case that $|\mathbf{w} - \mathbf{w}_n - \mathbf{w}_i| = \sin(\alpha - \alpha_n - \alpha_i)$ for some $i \in [n-1]$. After repeating this argument for $(n-1)$ times, we can assume, without loss of generality, that

$$|\mathbf{w}_1 + \dots + \mathbf{w}_i| = \sin(\alpha_1 + \dots + \alpha_i) \text{ for all } i \in [n].$$

Since $|\mathbf{w}_1 + \mathbf{w}_2| = \sin(\alpha_1 + \alpha_2)$, $|\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3| = \sin(\alpha_1 + \alpha_2 + \alpha_3)$ and D_1, D_2, D_3 cannot be covered by a cap with radius $< \alpha_1 + \alpha_2 + \alpha_3$, Lemma 5 implies that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are coplanar. Now let $\mathbf{w}'_2 := \mathbf{w}_1 + \mathbf{w}_2$ and D'_2 be the cap centered at $\mathbf{w}'_2/|\mathbf{w}'_2|$ with spherical radius $\alpha'_2 := \alpha_1 + \alpha_2$. One can check that D'_2 cover D_1, D_2 , and so D'_2, D_3, D_4 cannot be covered by a cap with radius $< \alpha'_2 + \alpha_3 + \alpha_4$. Since, in addition, $|\mathbf{w}'_2 + \mathbf{w}_3| = \sin(\alpha'_2 + \alpha_3)$ and $|\mathbf{w}'_2 + \mathbf{w}_3 + \mathbf{w}_4| = \sin(\alpha'_2 + \alpha_3 + \alpha_4)$, Lemma 5 implies that $\mathbf{w}'_2, \mathbf{w}_3, \mathbf{w}_4$ are coplanar, hence $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ are coplanar. Repeating this argument, we obtain that $\mathbf{w}_1, \dots, \mathbf{w}_n$ are coplanar. This reduces the problem for S^d to S^1 , and it is easy to check the equality condition for S^1 . \square

3 Other corollaries

The following two corollaries follow easily from Theorem 1. We call a pair of equal caps *an antipodal cap* if they are antipodal to each other.

Corollary 6. *Given a family of antipodal caps of radii r_1, \dots, r_n on the unit d -sphere, it is always possible to find a point common to them if the total radius $\sum_{i=1}^n r_i$ is at least $(n-1)\frac{\pi}{2}$.*

Proof. Apply Theorem 1 to the complement of the antipodal caps. \square

Corollary 7. *Given a family of great spheres C_1, \dots, C_n on the unit d -sphere, it is always possible to find an open cap of spherical radius $\pi/2n$ not intersecting any C_i .*

Proof. Apply Theorem 1 to the zones of width π/n with central great spheres C_1, \dots, C_n . \square

A spherical domain C is *centrally symmetric* with respect to a point c if for every $a \in C$ there is $b \in C$ such that c is the middle point of the shortest arc connecting a and b . The following corollary partially confirms Fejes Tóth's zone problem.

Corollary 8. *If a centrally symmetric convex spherical domain is covered with a set of zones of total width w , then it can be covered with one zone of width w .*

Proof. Suppose a convex spherical domain C is centrally symmetric with respect to c , and is covered by a set of zones P_1, \dots, P_n of total width w . Suppose D is the largest cap contained in C and centered at c . Let r be the spherical radius of D . Let D' be the equal cap that is antipodal to D . The complement of $D \cup D'$ is a zone of width $\pi - 2r$, which together with P_1, \dots, P_n covers the sphere. Theorem 1 says that $\pi - 2r + w \geq \pi$ or equivalently $2r \leq w$. Let D touch the boundary of C at a, b which are centrally symmetric with respect to c . One can check that the zone of width $2r$ with central great circle passing through c and perpendicular to arc ab covers C . \square

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